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REPRESENTATION-FINITE TRIVIAL EXTENSION ALGEBRAS

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Let A be a finite-dimensional basic connected associative algebra over an algebraically closed field, and $T(A)=A\ltimes DA$ its trivial extension by its minimal injective cogenerator. We prove that $T(A)$ is representation-finite of Cartan class Δ if and only if A is an iterated tilted algebra of Dynkin class Δ. The proof also yields a construction procedure for iterated tilted algebras of Dynkin type.

Introduction

Let A be a finite-dimensional basic connected algebra over an algebraically closed field, and denote by $T(A)=A\ltimes DA$ the trivial extension of A by its minimal injective cogenerator DA. Then $T(A)$ is self-injective and, in fact, symmetric. It is thus natural to ask for a description of the algebras A such that $T(A)$ is representation-finite. This problem was already considered by Müller [15], Green and Reiten [9], and Iwanaga and Wakamatsu [14] in the case where the square radical of A is equal to zero. It was shown by Tachikawa [18] that if A is hereditary and representation-finite, then $T(A)$ is again representation-finite (see also [19]). In [20], Yamagata proved that if $T(A)$ is representation-finite, then the ordinary quiver of A contains no oriented cycles. Later, Hughes and Waschbüsch [13] and Hoshino [12] proved that if A is a tilted algebra of Dynkin type Δ, then $T(A)$ is representation-finite of Cartan class Δ, and conversely, if $T(A)$ is representation-finite of Cartan class Δ, then there exists a tilted algebra B of Dynkin type Δ such that $T(A)\xrightarrow{\sim}B\ltimes DB$ (compare also [7], [17]). The purpose of the present paper is to prove the following theorem:

Theorem. *Let A be a finite-dimensional basic connected k-algebra and $T(A) = A \ltimes DA$ be its trivial extension. Then $T(A)$ is representation-finite of Cartan class Δ if and only if A is an iterated tilted algebra of Dynkin type Δ.*

This result was already obtained in the case where $\Delta = \mathbb{A}_n$ in [13], using the classification of the iterated tilted algebras of type $\mathbb{A}_n$ given in [1]. The proof presented here, however, does not use the classification results for iterated tilted algebras of Dynkin type. As a corollary, we obtain a simple construction procedure for all iterated tilted algebras of Dynkin type Δ starting from the tilted algebras of the same type.

1. Preliminaries

1.1. We shall let k denote a fixed algebraically closed field, and A a basic, connected, finite-dimensional k-algebra. By mod A will be meant the category of finite-dimensional right A-modules. The letter Q will be used to denote the ordinary quiver of A, and $i, j, \ldots$ to denote its vertices. The simple A-module corresponding to the vertex i of Q will be denoted by $S(i)$, and its projective cover (respectively, its injective hull) by $P(i)$ (respectively, $I(i)$). We shall freely use properties of the Auslander-Reiten sequences and the Auslander-Reiten quiver, for both finite-dimensional and locally finite-dimensional algebras, and refer for these to [8].

1.2. A module T_A is called a tilting module [10] if $\mathrm{Ext}^1_A(T,T)=0$, $\mathrm{Ext}^2_A(T,-)=0$ and there exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$, with T' and T'' direct sums of summands of T.

A finite-dimensional k-algebra B is called iterated tilted of type Δ [1] if there exists a sequence of algebras $A_0, A_1, \ldots, A_m = B$ where A_0 is a hereditary algebra having Δ as underlying graph of its ordinary quiver, and a sequence of tilting modules $(T^{(i)}_{A_i})_{0 \le i \le m-1}$ such that $\mathrm{End}\, T^{(i)}_{A_i} = A_{i+1}$ and, for every indecomposable A_{i+1}-module M, we have either $M \otimes_{A_{i+1}} T^{(i)} = 0$, or $\mathrm{Tor}_1^{A_{i+1}}(M, T^{(i)}) = 0$. If $m \le 1$, B is called a tilted algebra [10]. The iterated tilted algebras of type $\mathbb{A}_n$ were classified in [1]. A characterisation of the iterated tilted algebras of Dynkin type is given in [11].

1.3. Let $D = \mathrm{Hom}_k(-,k)$ denote the standard duality of mod A. The trivial extension of A by its minimal injective cogenerator ${}_A(DA)_A$ is defined to be the algebra $T = A \ltimes DA$ whose additive structure is that of the group $A \oplus DA$, with the multiplication defined by

$$(x,f)(y,g) = (xy, xg + fy)$$

for $x, y \in A$ and $f, g \in {}_A(DA)_A$. It is well-known that the trivial extension algebra T is self-injective, and in fact symmetric. Thus, a connected trivial extension algebra

T is representation-finite if and only if the stable part of its Auslander–Reien quiver is isomorphic to $\mathbb{Z}\Delta/G$, where Δ is a Dynkin diagram (the Cartan class of T), and G an admissible group of automorphisms of $\mathbb{Z}\Delta$ [16].

1.4. Consider the matrix algebra:

$$\hat{A}=\begin{bmatrix} \ddots & \ddots & & & & \\ & M_{m-1} & A_{m-1} & & & \\ & & M_m & A_m & & \\ & & & M_{m+1} & A_{m+1} & \\ & & & & \ddots & \ddots \end{bmatrix}$$

in which matrices have only finitely many non-zero entries, $A_m=A$, $M_m={}_A(DA)_A$ for all $m\in\mathbb{Z}$, all the remaining entries are zero, and multiplication is induced from the canonical maps $A\otimes_A DA\to DA$, $DA\otimes_A A\to DA$ and the zero map $DA\otimes_A DA\to 0$ [13]. Clearly, $\hat{A}$ is locally finite-dimensional [6]. The identity maps $A_m\to A_{m-1}$, $M_m\to M_{m-1}$ induce an automorphism ν of $\hat{A}$, and the orbit space $\hat{A}/\nu$ inherits the structure of a finite-dimensional k-algebra, isomorphic to $T=A\ltimes DA$. Moreover, ν induces an automorphism of the category mod $\hat{A}$ of finite-dimensional $\hat{A}$-modules, and the quotient of the Auslander–Reiten quiver $\Gamma_{\hat{A}}$ under the automorphism ν is a set of complete connected components of Γ_T [13].

Observe that any complete set $(e_i)_{1\le i\le n}$ of primitive orthogonal idempotents of A can be lifted to a complete set of primitive orthogonal idempotents $(e_{(i,m)})_{1\le i\le n, m\in\mathbb{Z}}$ of $\hat{A}$. Thus ν induces also an automorphism of the ordinary quiver $\hat{Q}$ of $\hat{A}$. A full connected subquiver Q' of $\hat{Q}$ is called a complete ν-slice of $\hat{Q}$ if its vertex set Q_0' consists of a complete set of representatives of the ν-orbits of $\hat{Q}_0$, and for any path $i=i_0\to i_1\to\cdots\to i_t=j$ such that $i,j\in Q_0'$, we have $i_s\in Q_0'$ for all $0<s<t$. Given a complete ν-slice Q' of $\hat{Q}$, the algebra

$$\mathrm{End}_{\hat{A}}\Big(\bigoplus_{(i,m)\in Q_0'} e_{(i,m)}\hat{A}\Big)$$

is called the algebra of Q', and denoted by Alg(Q'). Then, if the quiver Q of A has no oriented cycles, and B is a finite-dimensional k-algebra, the following assertions are equivalent [13]:

(i) $B\ltimes DB \xrightarrow{\sim} A\ltimes DA$,

(ii) $\hat{B}\xrightarrow{\sim}\hat{A}$,

(iii) $B\xrightarrow{\sim}\mathrm{Alg}(Q')$ for some complete ν-slice Q' of $\hat{Q}$.

1.5. For a sink i in the quiver of A, we define the ν-reflection of Q, σ_i^+Q to be the full subquiver of $\hat{Q}$ with vertex set $\{(j,0)\in\hat{Q}_0 \mid j\in Q_0, j\neq i\}\cup\{(i,1)\}$. A ν-reflection sequence of sinks $i_1,i_2,\dots,i_t$ is a sequence of vertices of $\hat{Q}$ such that i_s is a sink of $\sigma_{i_{s-1}}^+\cdots\sigma_{i_1}^+Q$ for $1\le s\le t$. ν-reflection sequences of sources are defined dually. Then the complete ν-slices of $\hat{Q}$ are exactly the repeated ν-reflections of Q [13].

For a sink i of Q, let T_i^+A denote the one-point extension algebra

$$T_i^+A = \begin{bmatrix} A & 0 \\ I(i)_A & k_i \end{bmatrix}$$

(where $k_i = \operatorname{End} S(i) \tilde{\to} k$), and S_i^+A be the quotient of T_i^+A by the two-sided ideal generated by the idempotent e_i corresponding to i. Then T_i^+A has a complete set $e_1, e_2, \dots, e_n, e_{i'}$ of $n+1$ primitive orthogonal idempotents, and the ordinary quiver of S_i^+A is obtained from that of T_i^+A by deleting the vertex corresponding to e_i. Similarly, starting with a source j of Q, we may define the algebras T_j^-A and S_j^-A. Also, $S_i^+A \tilde{\to} \operatorname{Alg}(\sigma_i^+Q)$ and $S_j^-A \tilde{\to} \operatorname{Alg}(\sigma_j^-Q)$ [13], thus we call S_i^+A and S_j^-A the reflections of A.

2. Reflections of iterated tilted algebras

2.1. We shall first recall some definitions. Let A be an algebra of finite global dimension, the Cartan matrix of A is defined to be the $n \times n$ matrix $C_A = [c_{ij}]$, where

$$c_{ij} = \dim_k \operatorname{Hom}_A(P(i), P(j))$$

and the Euler characteristic of A is defined to be the bilinear form on the Grothendieck group $K_0(A)$ of A with matrix $B_A = [b_{ij}]$ given by

$$b_{ij} = \sum_{l \geq 0} (-1)^l \dim_k \operatorname{Ext}_A^l(S(i), S(j))$$

(this sum is in fact finite due to our hypothesis on A). We shall let χ_A denote the associated quadratic form. It is easily seen that C_A is invertible and that its inverse equals the transpose B_A^t of B_A. We have the following lemma:

Lemma. *Let A be an algebra such that the quiver Q of A has no oriented cycles, and i be a sink of Q. Then the Cartan matrices of A and $A' = S_i^+A$ are congruent, that is, there exists a matrix $X \in \mathrm{GL}(n, \mathbb{Z})$ such that $C_{A'} = XC_AX^t$.*

Proof. We may assume, without loss of generality, that $i = 1$. Observe that, since Q has no oriented cycles, the global dimension of A is finite. Let $1'$ denote the vertex added to Q in the construction of T_1^+A and let $P'(1'), P'(2), \dots, P'(n)$ denote the non-isomorphic indecomposable projective A'-modules. By construction, we have $\operatorname{rad} P'(1')_{A'} = I(1)_A / \operatorname{soc} I(1)$.

If $i \neq 1$, $j \neq 1$, we have

$$\dim_k \operatorname{Hom}_{A'}(P'(i), P'(j)) = \dim_k \operatorname{Hom}_A(P(i), P(j)).$$

It is clear that

$$\dim_k \operatorname{Hom}_{A'}(P'(1'), P'(1')) = 1 = \dim_k \operatorname{Hom}_A(P(1), P(1)).$$

If $i \neq 1$, $i' \neq 1'$, we have

$$\mathrm{Hom}_{A'}(P'(1'), P'(i)) = 0 = \mathrm{Hom}_A(P(i), P(1)).$$

Finally, if $i \neq 1$, $i \neq 1'$, we have

$$\dim_k \mathrm{Hom}_{A'}(P'(i), P'(1')) = \dim_k \mathrm{Hom}_A(P(i), I(1))$$
$$= \dim_k \mathrm{Hom}_A(P(1), P(i)).$$

Hence, let $\boldsymbol{u} = -\mathbf{dim}\, I(1)_A$ and consider the matrix $X = [\boldsymbol{u}\ \boldsymbol{e}_2 \cdots \boldsymbol{e}_n]$ with $\boldsymbol{u}$ as first column, and $\boldsymbol{e}_i$ $(2 \leq i \leq n)$ the canonical basis vectors. Then $X \in \mathrm{GL}(n, \mathbb{Z})$ (in fact, X^2 is the identity matrix), and a straightforward calculation shows that $C_{A'} = XC_A X^t$.

2.2. Proposition. *Let A be an iterated tilted algebra of Dynkin type Δ, and i be a sink in the quiver Q of A. Then S_i^+A is also iterated tilted of type Δ.*

Proof. By [13], S_i^+A has a preprojective component $\mathscr{C}$ in its Auslander–Reiten quiver. If M is an indecomposable module in $\mathscr{C}$, then its dimension-vector $\mathbf{dim}\, M$ is a root of the quadratic form $\chi_{S_i^+A}$. Since the previous lemma shows that $\chi_{S_i^+A}$ is congruent to χ_A, and by [10], χ_A is congruent to the Tits form of a hereditary algebra of Dynkin type Δ, it follows that $\chi_{S_i^+A}$ has only finitely many roots. Hence $\mathscr{C}$ is a finite component of $\Gamma_{S_i^+A}$ and, by a theorem of Auslander [4], S_i^+A is representation-finite. In fact, S_i^+A is simply connected, since it satisfies the (S)-condition [5] (for, A being simply connected [2], the indecomposable injective A-module $I(i)_A$ has separated socle factor, hence the indecomposable projective S_i^+A-module $P'(i')$ has separated radical. It is clear that the other indecomposable projective S_i^+A-modules have separated radicals). Now, it is proved in [11] that a simply connected algebra is iterated tilted of Dynkin type Δ provided the quadratic form associated to its Euler characteristic is congruent to the Tits form of a hereditary algebra of Dynkin type Δ. This shows that S_i^+A is indeed iterated tilted of Dynkin type Δ.

Dually, if j is a source of Q, S_j^-A is also iterated tilted of type Δ. Inductively, if $i_1, i_2, \dots, i_t$ is a ν-reflection sequence of sinks (respectively, if $j_1, j_2, \dots, j_s$ is a ν-reflection sequence of sources), then $S_{i_t}^+ \cdots S_{i_1}^+A$ (respectively, $S_{j_s}^- \cdots S_{j_1}^-A$) is also iterated tilted of type Δ.

3. The main result

3.1. Theorem. *Let A be a basic, connected finite-dimensional k-algebra, and let $T = A \ltimes DA$ be its trivial extension by its minimal injective cogenerator. Then T is representation-finite of Cartan class Δ if and only if A is an iterated tilted algebra of Dynkin type Δ.*

Proof. Let A be an iterated tilted algebra of Dynkin type Δ, we shall prove that

$T = A \ltimes DA$ is representation-finite of type Δ. Since A is simply connected [2], its ordinary quiver Q must contain at least one strong sink i (that is to say, a vertex $i \in Q_0$ such that there is no chain of indecomposable modules and irreducible maps $I(j)_A \to \cdots \to I(i)_A$ for any $j \in Q_0$). A strong sink is always a sink [13].

Clearly, there are canonical embeddings of $\operatorname{mod} A$ and $\operatorname{mod}(S_i^+ A)$ into $\operatorname{mod}(T_i^+ A)$, where the first category can be described as the class of $T_i^+ A$-modules without the simple injective $I(i')_{T_i^+ A}$ as a composition factor, while the second is the class of modules without the simple projective $P(i)_{T_i^+ A}$ as a composition factor. Moreover, the only $T_i^+ A$-module which is not in either one of these classes is the projective-injective module $P(i')_{T_i^+ A} = I(i)_{T_i^+ A}$ [13]. Since, by Proposition 2.2, $S_i^+ A$ is also iterated tilted of type Δ, we may iterate this process to obtain the Auslander–Reiten quiver of $\hat{A}$. Let $i_1, i_2, \ldots, i_t$ be a strong ν-reflection sequence of sinks, that is to say, a sequence such that i_r is a strong sink in the quiver $\sigma_{i_{r-1}}^+ \cdots \sigma_{i_1}^+ Q$ of the algebra $S_{i_{r-1}}^+ \cdots S_{i_1}^+ A$ (for all $1 \le r \le t$), then the algebra $S_{i_r}^+ \cdots S_{i_1}^+ A$ is iterated tilted of type Δ, and a straightforward induction yields the Auslander–Reiten quiver of $T_{i_t}^+ \cdots T_{i_1}^+ A$. Again, any indecomposable $T_{i_t}^+ \cdots T_{i_1}^+ A$-module is either projective-injective, or else can be identified with an indecomposable module over one of the iterated tilted algebras $S_{i_r}^+ \cdots S_{i_1}^+ A$ $(1 \le r \le t)$. This procedure yields in fact a connected component Γ of the Auslander–Reiten quiver of $\hat{A}$. Indeed, we construct inductively strong ν-reflection sequence of sinks $(i_t)_{t \ge 1}$ and of sources $(j_s)_{s \ge 1}$, then every vertex of $\hat{Q}$ belongs to $\bigcup_{t=0}^{\infty} \sigma_{i_t}^+ \cdots \sigma_{i_1}^+ Q$ or to $\bigcup_{s=0}^{\infty} \sigma_{j_s}^- \cdots \sigma_{j_1}^- Q$ and the categories $\operatorname{mod}(T_{i_t}^+ \cdots T_{i_1}^+ A)$, and $\operatorname{mod}(T_{j_s}^- \cdots T_{j_1}^- A)$ embed naturally in $\operatorname{mod} \hat{A}$. Thus, as in [13], we obtain a connected component Γ of $\Gamma_{\hat{A}}$, containing all projective $\hat{A}$-modules and in which all modules have bounded composition length. It then follows from [4] that $\Gamma = \Gamma_{\hat{A}}$. On the other hand, the orbit space $\Gamma_{\hat{A}}/\nu$ is a connected component of Γ_T, which contains all the projective T-modules, and in which all modules have bounded length. Therefore $\Gamma_{\hat{A}}/\nu$ is actually equal to Γ_T, and T is representation-finite.

We next show that the Cartan class Δ' of T is equal to Δ. Indeed, T being a representation-finite trivial extension algebra, there exists a tilted algebra B of type Δ' such that $T \xrightarrow{\sim} B \ltimes DB$ ([12],[13]). However, $A \ltimes DA \xrightarrow{\sim} B \ltimes DB$ implies that B is the algebra of a complete ν-slice Q' of $\hat{Q}$, which must be in fact a repeated ν-reflection of Q. More precisely, there exists a ν-reflection sequence of sinks $i_1, i_2, \ldots, i_t$ (or of sources $j_1, j_2, \ldots, j_s$) of Q such that $Q' = \sigma_{i_t}^+ \cdots \sigma_{i_1}^+ Q$ resp., $Q' = \sigma_{j_s}^- \cdots \sigma_{j_1}^- Q$) and therefore $B \xrightarrow{\sim} S_{i_t}^+ \cdots S_{i_1}^+ A$ (resp., $B \xrightarrow{\sim} S_{j_s}^- \cdots S_{j_1}^- A$). This indeed follows by induction on t (resp., s) and the fact that, for a sink i, (resp., a source j) of Q, $S_i^+ A \xrightarrow{\sim} \operatorname{Alg}(\sigma_i^+ Q)$ (resp., $S_j^- A \xrightarrow{\sim} \operatorname{Alg}(\sigma_j^- Q)$). However, A being iterated tilted of type Δ, Proposition 2.2 implies that B is of the same type, hence $\Delta = \Delta'$.

We have thus proved that if A is iterated tilted of Dynkin type Δ, its trivial extension algebra $A \ltimes DA$ is representation-finite of Cartan class Δ. Conversely, if $A \ltimes DA$ is representation-finite of Cartan class Δ, there exists a tilted algebra B of type Δ such that $B \ltimes DB \xrightarrow{\sim} A \ltimes DA$ [13]. Using the same argument as above, we can find a ν-reflection sequence of sinks $i_1, i_2, \ldots, i_t$ (or of sources $j_1, j_2, \ldots, j_s$) of

Q such that $A \xrightarrow{\sim} S_{i_t}^+ \cdots S_{i_1}^+ B$ (resp., $A \xrightarrow{\sim} S_{j_s}^- \cdots S_{j_1}^- B$). Applying again Proposition 2.2, it follows that A is iterated tilted of type Δ.

3.2. We have the following obvious corollary:

Corollary. *Let A be an iterated tilted algebra of Dynkin type Δ. Then there exists a tilted algebra B of type Δ, and a ν-reflection sequence of sinks $i_1, i_2, \ldots, i_t$ in the quiver of B such that $A \xrightarrow{\sim} S_{i_t}^+ \cdots S_{i_1}^+ B$.*

3.3. Remark. Using the classification of the iterated tilted algebras of types $\mathbb{B}_n$ and $\mathbb{C}_n$ [2], it is shown in [3] that the main theorem holds in these cases too. Finally, an easy computation shows that it holds as well in the cases $\mathbb{F}_4$ and $\mathbb{G}_2$.

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